RESEARCH NOTE

An Application of Fuzzy Set Representation

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In this note a more general version of fuzzy representation theorem considered by D.A. Ralescu (1975, 1992) is presented. Then a fuzzy algebraic application of this theorem is given.

INTRODUCTION

Representation theorem for L-fuzzy sets has been given by Negoita and Ralescu in [1], where L is a complete lattice with an extra condition called L2. They have also stated many applications of this theorem in diverse fields [2,3]. This notion has been studied by other researchers as well [4–6] and is used to characterize fuzzy algebraic structures in [7]. Recently Ralescu in [8] has given a generalized version of this representation where L is the interval $[0,1]$. Here it is shown that the original representation theorem in [1] is true for any complete lattice without any more condition and other results of [8] are extendible under some conditions. Then, a generalization of a result given in [7] for $\Omega\wedge$-fuzzy subgroups ($\wedge$-fuzzy subpolygroup) of a group (polygroup) is presented . The representation given here makes it possible to characterize many fuzzy concepts whose valuations are in a lattice rather than the interval $[0,1]$.

FUZZY SET REPRESENTATION

Let $L = (L, \leq, \vee, \wedge)$ be a complete lattice with the least element 0 and the greatest element 1. By a fuzzy set A the map $A : X \to L$ is meant. For any $a \in L$, the a-level of A is defined as:

$$A_a = \{x \in X : A(x) \geq a\} .$$

Definition 1

Let X be a nonempty set, by a closure set system on X, a set Σ of subsets of X which is closed under arbitrary intersection is meant, i.e. $\cap_{a\in M} B_a \in \Sigma$ for all index set M and any $B_a \in \Sigma$.

The above definition implies that $X \in \Sigma$ for any closure system Σ on X, if for the empty set $\emptyset$ the usual assumption is imposed:

$$\cap_{a\in\emptyset} B_a = X .$$

The idea of considering fuzzy sets whose a-levels belong to a closure system appears in [1,7].

Definition 2

Let Σ be a set of subsets of the nonempty set X. The fuzzy set A is said to be a fuzzy Σ-subset of X if and only if,

$$A_a \in \Sigma,\ \text{for all } a \in L .$$

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Note 1

Let $\{B_a\}_{a\in L}$ be a class of subsets of X such that for any subset $M \subseteq L$,

$$B_{\vee_{a\in M}} = \cap_{a\in M} B_a \ . \tag{1}$$

From Statement 1, it is then obvious that if $b, g \in L$, the following is obtained:

$$b \leq g \Longrightarrow B_g \subseteq B_b \ . \tag{2}$$

Also $\{B_a\}_{a\in L}$ is a closure set system on X, therefore, there is $b \in L$ such that $X = B_b$. However, from Statement 2, the following is obtained:

$$B_a \subseteq B_0, \text{ for all } a \in L \ . \tag{3}$$

So, in particular, from Statement 3 $X = B_b \subseteq B_0$ is obtained. Hence:

$$B_0 = X. \tag{4}$$

Now the following representation theorem could be proved by the aid of Note 1, and later this theorem will be also used to generalize Theorem 2.4 of [7] as in Corollary 3.

Theorem 1 (Fuzzy Set Representation)

Let $\{B_a\}_{a\in L}$ be a class of subsets of X. The necessary and sufficient condition that there is a fuzzy set A for which $A_a = B_a$, for all $a \in L$ is that $B_{\vee_{a\in M}} = \cap_{a\in M} B_a$, for all $M \subseteq L$.

Proof

The necessary part is easy. So only the proof of the other part is given. Now suppose that Statement 1 holds. Define a fuzzy set $A : X \to L$ as follows:

$$A(x) = \vee_{x\in B_a} a \ .$$

Now let $b \in L$ be arbitrary. Suppose $x \in A_b$. Then, from definitions of A_b and $A(x)$, the following is obtained:

$$b \leq \vee_{x\in B_a} a \ . \tag{5}$$

Therefore using Statements 2 and 5, the following is obtained:

$$B_{\vee_{x\in B_a} a} \subseteq B_b \ . \tag{6}$$

Likewise, using Statements 6 and 1, Statement 7 is obtained:

$$\cap_{x\in B_a} B_a \subseteq B_b \ . \tag{7}$$

It is obvious that $x \in \cap_{x\in B_a} B_a$. Therefore, using Statement 7, $x \in B_b$. Hence:

$$A_b \subseteq B_b, \text{ for all } b \in L \ . \tag{8}$$

Now suppose that $x \in B_b$. Then, Statement 5 holds by definition of $\vee$. Moreover, from definitions of $A(x)$ and Statement 5, it is concluded that $b \leq A(x)$. That is $x \in A_b$. Thereby,

$$B_b \subseteq A_b, \text{ for all } b \in L \ . \tag{9}$$

Therefore, using Statements 8 and 9, the following is obtained:

$$A_a = B_a, \text{ for all } a \in L \ . \tag{10}$$

Theorem 1 is a relaxed version of Lemma 1 of [1], in the sense that condition L2 is not imposed anymore, where L2 is defined by Statement 12. However, in order to generalize Theorem 2 of [8] some conditions on the lattice L need to be imposed, which is stated below.

From now on it is always assumed that L' is a lattice L, which has the following extra properties:

L1) If $\{a_i\}_{i\in\Omega} \subseteq L'$ is such that $\bigvee_{i\in\Omega} a_i = a \in L'$, then there is an increasing subsequence of $\{a_i\}_{i\in\Omega}$, say $a_1 \leq a_2 \leq \cdots$, such that:

$$\bigvee_{n=1}^{\infty} a_n = a \ . \tag{11}$$

L2) For all

$$Y \subseteq L', a < \vee_{y\in Y} y \Longrightarrow \text{ there is } b \in Y \text{ such that } a \leq b. \tag{12}$$

Now a generalized version of Theorem 1 can be given as follows:

Theorem 2

Let $\phi : L' \to L'$ be given and let $\{B_a\}_{a\in L'}$ be a class of subsets of X. The necessary and sufficient conditions that there is a fuzzy set A for which $A_{\phi(a)} = B_a$, for all $a \in L'$ are that:

1. $\phi(a) \leq \phi(b) \Longrightarrow A_b \subseteq A_a$,
2. $\phi(a_1) \leq \phi(a_2) \leq \cdots$ and $\bigvee_{n=1}^{\infty} \phi(a_n) = \phi(a) \Longrightarrow \bigcap_{n=1}^{\infty} A_{a_n} = A_a$.

Proof

The necessary part is easy. So only the proof of the other part is given. Define $A(x)$ as:

$$A(x) = \vee_{x\in B_a}\phi(a) \ .$$

Now, by using Statements 11 and 12 exactly the same line of the proof given in [8] will work here.

Definition 3

A unary operation $C : L \to L$ is called a complement on L [9,10] if,

1. $x \leq y \Longrightarrow C(y) \leq C(x)$, for all $x, y \in L$,
2. $C(C(x)) = x$, for all $x \in L$.

It is obvious that one can find that $C(0) = 1$.

As a special case of Theorem 2, if ϕ is chosen as the identity function or a complement, then the following results are obtained respectively.

Corollary 1

Let $\{B_a\}_{a\in L'}$ be a class of subsets of X. The necessary and sufficient conditions that there is a fuzzy set A for which $A_a = B_a$, for all $a \in L'$ are that:

1. $a \leq b \Longrightarrow A_b \subseteq A_a$,
2. $a_1 \leq a_2 \leq \cdots$ and $\bigvee_{n=1}^{\infty} a_n = a \Longrightarrow \bigcap_{n=1}^{\infty} A_{a_n} = A_a$.

Corollary 2

Let $\{B_a\}_{a\in L'}$ be a class of subsets of X and assume there is a complement C on L'. The necessary and sufficient conditions that there is a fuzzy set A for which $A_a = B_a$, for all $a \in L'$ are that:

1. $b \leq a \Longrightarrow A_b \subseteq A_a$,
2. $\cdots \leq a_n \leq \cdots \leq a_2 \leq a_1$ and $\bigwedge_{n=1}^{\infty} a_n = a \Longrightarrow \bigcap_{n=1}^{\infty} A_{a_n} = A_a$.

The following is a generalization of Theorem 3.4 of [7].

Corollary 3

Let Σ be a closure set system on X. If $\{B_a\}_{a\in L}$ is a subclass of Σ, then the fuzzy set $A : X \to L$ defined by $A(x) = \vee_{x\in B_a} a$ is a fuzzy Σ-subset. Conversely every fuzzy Σ-subset A of X could be obtained as above.

Proof

From Theorem 1, it is known that Statement 10 holds. Therefore, A is a fuzzy Σ-subset and from Statement 4 it is obvious that $A_0 = B_0 = X \in \Sigma$. The proof of the other part is exactly the same as Theorem 3.4 of [7].

Remark 1

1. Suppose G is an Ω-group as in Definition 7. Let:

   $$\Sigma = \{H : H \text{ is an } \Omega\text{-subgroup of } G \text{ as in Definition 8 }\} .$$

2. Suppose pG is a polygroup $(pG, .)$ as in Definition 5. Let:

   $$\Sigma' = \{H : H \text{ is a subpolygroup of } pG \text{ as in Definition 8 }\} .$$

   Then, it is easy to see that Σ and Σ' are closure set systems on G and pG, respectively.

APPLICATION

In this section an application of Theorem 1 is given which generalizes some concepts in Section 4 of [7], i.e. characterization of many fuzzy algebraic structures, such as fuzzy(normal) groups, fuzzy modules, fuzzy vector spaces etc.

Some definitions required here are stated from [10–12]. Let G be a group with identity element e.

Definition 4

An Lt-norm T is a binary operation $T : L \times L \rightarrow L$ having the properties:

1. $T(x, 1) = x$,
2. $T(x, y) = T(y, x)$,
3. $T(x, y) \leq T(u, y)$ if $x \leq u$,
4. $T(x, T(y, z)) = T(T(x, y), z))$.

Note that $\wedge$ is actually an Lt-norm. One can find Lt-norms other than $\wedge$ (for such examples see [11,13]).

Let H be a nonempty set and $P^*(H) = p(H)\backslash\emptyset$, where $P(H)$ denotes the power set of H.

Definition 5

$(H, .)$ is called a polygroup if $. : H \times H \rightarrow P^*(H)$ is a map such that the following conditions hold:

1. $x.(y.z) = (x.y).z$ for all $x, y, z \in H$,
2. There exists $i \in H$, called the neutral element of H, such that $i.x = x.i = \{x\}$, for all $x \in H$,
3. For all $x \in H$ there exists a unique $x' \in H$ such that $i \in x.x' \cap x'.x$. x' is said to be the opposite of x and is denoted by x,
4. For all $x, y, z \in H$,

$$z \in x.y \Longrightarrow x \in z.y^{-1} \Longrightarrow y \in x^{-1}.z \ .$$

Moreover, let $K \subseteq H$. Then K is called a subpolygroup of H if $i \in K$ and $(K, .)$ is itself a polygroup.

Definition 6

Let T be an Lt-norm. A fuzzy set $A : X \rightarrow L$ is called a T-fuzzy subgroup of G if:

1. $A(xy) \geq T(A(x), A(y))$, for all $x, y \in G$,
2. $A(x^{-1}) = A(x)$, for all $x \in G$,
3. $A(e) = 1$.

Definition 7

The order pair $(G, *)$, where $* : G \times \Omega \rightarrow G$, $(a, \omega) \mapsto a * \omega$ has the property:

$$ab * \omega = (a * \omega)(b * \omega), \text{ for all } a, b \in G,$$
$$\text{for all } \omega \in \Omega \ ,$$

is called an operator group or Ω-group. Ω is called an operator domain and each $\omega \in \Omega$ is called an operator.

Definition 8

Let G be an Ω-group and H a subgroup of G. H is called an Ω-subgroup of G if H is Ω-admissible, i.e.,

$$h * \omega \in H, \text{ for all } h \in H, \text{ for all } \omega \in \Omega \ .$$

Definition 9

Let A be a fuzzy subset of Ω-group G. Then A is called:

1. Ω-admissible on G, if:
$$A(x) \leq A(x * \omega), \text{ for all } x \in G,$$
$$\text{for all } \omega \in \Omega \ .$$
2. ΩT-fuzzy subgroup or $T\Omega$-admissible on G, if A is both Ω-admissible and T-fuzzy subgroup.

Definition 10

Let $(pG, .)$ be a polygroup and T be an Lt-norm. A fuzzy subset $A : pG \rightarrow L$ is called a T-fuzzy subpolygroup, if

1. $A(z) \geq T(A(x), A(y))$, for all $z \in x.y$ and all $x, y \in pG$,
2. $A(x^{-1}) = A(x)$, for all $x \in pG$,
3. $A(i) = 1$, where i is the neutral element of $(pG, .)$.

The following lemma is easy to prove. For the polygroup case also see Theorem 4.5 of [14].

Lemma 1

The fuzzy set A is an $\Omega\wedge$-fuzzy subgroup (a $\wedge$-fuzzy polygroup) of $G(pG)$ if and only if all nonempty a-level A_a of A are Ω-subgroup (polygroup) of $G(pG)$.

Theorem 3

Let $\{B_a\}_{a\in L}$ be a class of Ω-subgroups (polysubgroups) of $G(pG)$. The necessary and sufficient condition that there is a ΩT-fuzzy subgroup (T-fuzzy subpolygroup) A of $G(pG)$ for which $A_a = B_a$, for all $a \in L$ is that $B_{\vee_{a\in M}} = \cap_{a\in M} B_a$ for all $M \subseteq L$, where $T = \wedge$.

Proof

The proof follows from Theorem 1 and Lemma 1.

A very general version of Theorem 3, the proof of which follows from Lemma 1 and Theorem 2, is as follows.

Theorem 4

Let $\phi : L' \rightarrow L'$ be given, and $\{B_a\}_{a\in L'}$ be a class of Ω-subgroups (polygroups) of $G(pG)$. The necessary and sufficient conditions that there is ΩT-fuzzy subgroup (T-fuzzy subpolygroup) A of $G(pG)$ for which $A_{\phi(a)} = B_a$, for all $a \in L'$ are that:

1. $\phi(a) \leq \phi(b) \Longrightarrow A_b \subseteq A_b$,
2. $\phi(a_1) \leq \phi(a_2) \leq \cdots$ and $\bigvee_{n=1}^{\infty} \phi(a_n) = \phi(a) \Longrightarrow \bigcap_{n=1}^{\infty} A_{a_n} = A_a$,

where $T = \wedge$.

More conclusions are drawn from Corollary 3 if the class $\{B_a\}_{a\in L}$ are Ω-subgroups (polygroups) of $G(pG)$, and $T = \wedge$, by the aid of Remark 1.

This section ends with an open question.

Question 1

Is there any characterization (i.e. any version of Theorem 3 or Theorem 2) of $T\Omega$-fuzzy subgroup or T-fuzzy subpolygroup, when T is arbitrary?

Remark 2

The author found, after completing this paper, that the notion of an Lt-norm and a version of Theorem 1 are also studied in Lemma 1 of [12].

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