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Regular $\mathcal{RGC}_n$-Commutative Semigroups

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In this paper, it is proven that a semigroup is regular and $\mathcal{RGC}_n$-commutative if, and only if, it is a spined product of a commutative Clifford semigroup and a right regular band.

INTRODUCTION

A semigroup S is said to be an $\mathcal{R}$-commutative semigroup if, for every couple $(a,b) \in S \times S$, there is an element, $u \in S^1$, such that $ab = bau$ (see [1,2]). The $\mathcal{R}$-commutativity is not hereditary for subsemigroups, in general. The next lemma gives a sufficient condition for an ideal K of a semigroup S when the $\mathcal{R}$-commutativity of S is hereditary for K.

Lemma 1

If K is an ideal of an $\mathcal{R}$-commutative semigroup, such that K is simple, then K is $\mathcal{R}$-commutative [1].

A semigroup S is called a conditionally commutative semigroup if, for every $a, b \in S, ab = ba$ implies $axb = bxa$ for all $x \in S$ (see for example [3]). It is clear that every conditionally commutative semigroup satisfies the identity $axa^2 = a^2xa$.

In [4], B. Pondělíček defined the notion of the generalized conditionally commutative semigroup (or $\mathcal{GC}$-commutative semigroup) as a semigroup satisfying the identity $axa^2 = a^2xa$. He proved that a $\mathcal{GC}$-commutative semigroup satisfies the identity $axa^i = a^ixa$ for every positive integer $i (\geq 2)$.

For a positive integer n a semigroup is called a generalized conditionally n-commutative semigroup (or $\mathcal{GC}_n$-commutative semigroup) if it satisfies the identity $a^nxa^i = a^ixa^n$, for every integer $i \geq 2$ (see [5]). It is noted that the $\mathcal{GC}_1$-commutative semigroups are the $\mathcal{GC}$-commutative ones and the conditionally commutative semigroups are $\mathcal{GC}_n$-commutative for every positive integer n.

An $\mathcal{R}$-commutative and $\mathcal{GC}_n$-commutative semigroup is called an $\mathcal{RGC}_n$-commutative semigroup (see [5]).

An element a of a semigroup S is called regular, if there exists x in S, such that $axa = a$. The semigroup S is called regular, if all its elements are regular.

In this paper, the regular $\mathcal{RGC}_n$-commutative semigroups are described. For the notions not defined here, please refer to [3,6,7].

REGULAR $\mathcal{R}$-COMMUTATIVE SEMIGROUPS

A semigroup S is called an orthogroup, if it is a union of its subgroups and the set E_S of all idempotent elements of S is a subsemigroup of S.

A semigroup S is called a rectangular group, if it is a direct product of a rectangular band B and a group G. In particular, if B is a right zero semigroup, then S is called a right group; if G is commutative and B is a right zero semigroup, then S is called a right abelian group.

Lemma 2

A semigroup is an orthogroup if and only if it is a semilattice of rectangular groups (see [8]).

Theorem 1

Every regular $\mathcal{R}$-commutative semigroup is an orthogroup, which is a semilattice of right groups.

Proof

Let S be a regular $\mathcal{R}$-commutative semigroup. Then, for every element $a \in S$, there are elements $x \in S$ and $y \in S^1$, such that $a = a(xa) = a(axy) = a^2xy$. Thus S is also right regular. By Theorem 4.3 of [7], S is a union of groups. Since S is $\mathcal{R}$-commutative, then, for every idempotent elements e and f of S, there are elements $x, y \in S^1$, such that $ef = fex$ and $fe = efy$. Then:

$$(ef)^2 = e(fe)f = e(efy)f = (efy)f = f(ef)$$

$$= f(fex) = fex = ef,$$

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that is, the set of all idempotent elements of S is a subsemigroup. Consequently, S is an orthogroup. By Lemma 2, S is a semilattice Y of rectangular groups $S_\alpha(\alpha \in Y)$. Let $\alpha \in Y$ be an arbitrary element. Let $A_\alpha = \cup\{S_\beta : \alpha \leq \beta\}$. If $b, c \in A_\alpha$ be arbitrary elements, say $b \in S_\beta$ and $c \in S_\gamma(\alpha \leq \beta, \gamma)$, then, $\alpha \leq \beta\gamma$ and so $bc, cb \in A_\alpha$. Thus A_α is a subsemigroup of S. It is clear that bc and cb are in the same $S_\delta(\alpha \leq \delta)$. As S is $\mathcal{R}$-commutative, there is an element $x \in S^1$, such that $bc = cbx$. If $x \in S_\xi$, then $\delta = \delta\xi$ and so $\delta \leq \xi$, which implies $\alpha \leq \xi$. Thus $x \in S_\xi \subseteq A_\alpha$. Hence, A_α is $\mathcal{R}$-commutative. Since S_α is an ideal of A_α, and S_α is simple, then, by Lemma 1, it follows that S_α is $\mathcal{R}$-commutative. If $S_\alpha = L_\alpha \times G_\alpha \times R_\alpha$, where L_α is a left zero semigroup, R_α is a right zero semigroup and G_α is a group, then for arbitrary elements, $l_1, l_2 \in L_\alpha, r \in R_\alpha$, and the identity element e of G_α, there is an element $x \in S_\alpha$ such that:

$$(l_1, e, r) = (l_1 l_2, e, r) = (l_1, e, r)(l_2, e, r)$$

$$= (l_2, e, r)(l_1, e, r)x,$$

from which one can conclude that $l_1 = l_2$. Thus, L_α has only one element and, so, S_α is a right group.■

REGULAR $\mathcal{RGC}_n$-COMMUTATIVE SEMIGROUPS

By Theorem 1, one can formulate a corollary about regular $\mathcal{RGC}_n$-commutative semigroups.

Corollary 1

Every regular $\mathcal{RGC}_n$-commutative semigroup is an orthogroup, which is a semilattice of right abelian groups.

Proof

By the previous theorem, a regular $\mathcal{R}$-commutative semigroup is an orthogroup, such that it is a semilattice Y of right groups $G_\alpha \times R_\alpha$, where G_α are groups, R_α are right zero semigroups, $\alpha \in Y$. If $h, g \in G_\alpha, r \in R_\alpha, \alpha \in Y$ are arbitrary elements, then;

$$(h^n g h^{n+1}, r) = (h, r)^n (g, r)(h, r)^{n+1}$$

$$= (h, r)^{n+1}(g, r)(h, r)^n$$

$$= (h^{n+1} g h^n, r).$$

Thus, $gh = hg$ and, so, G_α is an abelian group.■

In the investigations, notations of the Preston's Theorem will be used, which gives a construction for orthogroups and so it is formulated in the next lemma.

Lemma 3 (Preston's Theorem [8])

Let E be a band and let $E = \cup_{\alpha \in Y} E_\alpha$ be the decomposition of E into a semilattice Y of rectangular bands $E_\alpha = L_\alpha \times R_\alpha(\alpha \in Y)$. For each α in Y, let G_α be a group, 1_α be the identity element of G_α, $S_\alpha = L_\alpha \times G_\alpha \times R_\alpha$ and $S = \cup_{\alpha \in Y} S_\alpha$. Identify $1_\alpha \times E_\alpha$ with E_α.

For each pair of elements $\alpha, \beta \in Y$ with $\alpha > \beta$, let $\psi_{\alpha,\beta}$ be a homomorphism of G_α into G_β and let $t_{\alpha,\beta}$ $(\tau_{\alpha,\beta})$ be a left (right) representation of S_α by transformations of L_β (R_β) such that if $e_\alpha = (i_\alpha, \kappa_\alpha) \in E_\alpha$, and $(j_\beta, \lambda_\beta) \in E_\beta$, then:

$$e_\alpha f_\beta = ((t_{\alpha,\beta} e_\alpha) j_\beta, \lambda_\beta),$$

$$f_\beta e_\alpha = (j_\beta, \lambda_\beta(e_\alpha \tau_{\alpha,\beta})).$$

Define $\psi_{\alpha,\alpha}, t_{\alpha,\alpha}$ and $\tau_{\alpha,\alpha}$ $(\alpha \in Y)$ as follows. Let $\psi_{\alpha,\alpha}$ be the identity automorphism of G_α. For $A = (i_\alpha, a_\alpha, \kappa_\alpha) \in S_\alpha$, let $t_{\alpha\alpha} A$ map every element of L_α onto i_α, and let $A\tau_{\alpha,\alpha}$ map every element of R_α onto κ_α.

Define the product AB of any two elements $A, B \in S$, as follows. Suppose $A = (i_\alpha, a_\alpha, \kappa_\alpha) \in S_\alpha$ and $B = (j_\beta, b_\beta, \lambda_\beta) \in S_\beta$. Let $\gamma = \alpha\beta$ (product in Y), and let:

$$(k_\gamma, \mu_\gamma) = (i_\alpha, \kappa_\alpha)(j_\beta, \lambda_\beta),$$

be the given product of $(i_\alpha, \kappa_\alpha)$ and (j_β, λ_β) in the band E. Then, define:

$$AB = ((t_{\alpha,\gamma} A) k_\gamma, (a_\alpha \psi_{\alpha,\gamma})(b_\beta \psi_{\beta,\gamma}), \mu_\gamma(B\tau_{\beta,\gamma})).$$

This definition is consistent with the given products in E and the various S_α $(\alpha \in Y)$. When $\alpha \geq \beta$, the product AB simplifies to:

$$AB = ((t_{\alpha,\beta} A) j_\beta, (a_\alpha \psi_{\alpha,\beta}) b_\beta, \lambda_\beta),$$

$$BA = (j_\beta, b_\beta(a_\alpha \psi_{\alpha,\beta}), \lambda_\beta(A\tau_{\alpha,\beta})).$$

Assume, furthermore, that the following conditions hold for all $\alpha, \beta, \gamma \in Y$, such that $\alpha > \beta > \gamma$ and, for all $A \in S_\alpha$, $B \in S_\beta$:

$$\psi_{\alpha,\beta}\psi_{\beta,\gamma} = \psi_{\alpha,\gamma},$$

$$t_{\beta,\gamma}(AB) = (t_{\alpha,\gamma} A)(t_{\beta,\gamma} B),$$

$$t_{\beta,\gamma}(BA) = (t_{\beta,\gamma} B)(t_{\alpha,\gamma} A),$$

$$(AB)\tau_{\beta,\gamma} = (A\tau_{\alpha,\gamma})(Bt_{\beta,\gamma}),$$

$$(BA)\tau_{\beta,\gamma} = (B\tau_{\beta,\gamma})(A\tau_{\alpha,\gamma}).$$

Then S becomes an orthogroup and, conversely, every orthogroup can be constructed in this way.■

Let S_1 and S_2 be semigroups having Y as their common greatest semilattice homomorphic image. Let $\phi_1 : S_1 \mapsto Y$ and $\phi_2 : S_2 \mapsto Y$ be the canonical homomorphisms. Let $S = \{(a,b) \in S_1 \times S_2 : \phi_1(a) = \phi_2(b)\}$. S is a subdirect product of S_1 and S_2 which is called the spined product of S_1 and S_2.

A Clifford semigroup, means a regular semigroup S, in which the idempotent elements are central, i.e. $es = se$ for every idempotent element e and every s in S. It is well known that a semigroup is a Clifford semigroup if and only if it is a (strong) semilattice of groups (see, for example, [6]).

A band is called a right regular band if it satisfies the identity $ab = bab$. Since every band is a semilattice of rectangular bands, then it is easy to see that a band is right regular if and only if it is a semilattice of right zero semigroups.

Theorem 2

A semigroup is regular and $\mathcal{RGC}_n$-commutative if, and only if it is a spined product of a commutative Clifford semigroup and a right regular band.

Proof

Let S be a regular $\mathcal{RGC}_n$-commutative semigroup. Then, by Corollary 1, S is an orthogroup, which is a semilattice Y of right abelian groups $G_\alpha \times R_\alpha (\alpha \in Y)$. It is clear that $E_S = \cup_{\alpha \in Y}(e_\alpha \times R_\alpha)$, where e_α denotes the identity of G_α. On $R = \cup_{\alpha \in Y} R_\alpha$, one can define an operation $*$ as follows: If $r_\alpha \in R_\alpha$ and $r_\beta \in R_\beta$ be arbitrary elements and $(e_\alpha, r_\alpha)(e_\beta, r_\beta) = (e_{\alpha\beta}, r_{\alpha\beta})$, for some $r_{\alpha\beta} \in R_{\alpha\beta}$, then, let:

$$r_\alpha * r_\beta = r_{\alpha\beta}.$$

It is easy to see that $(R, *)$ is a semigroup, which is a semilattice, Y, of the right zero semigroups R_α $(\alpha \in Y)$. Thus $(R, *)$ is a right regular band. It is noted that $(e_\alpha, r_\alpha) \to r_\alpha$ is an isomorphism of E_S onto $(R, *)$. By Preston's Theorem, the product in S is determined by right representations $()\tau_{\alpha,\beta}$ of S_α, by transformations of R_β and homomorphism $()\psi_{\alpha,\beta}$ of G_α into G_β $(\alpha, \beta \in Y$, with $\alpha \geq \beta)$. It is clear that $\{\psi_{\alpha,\beta}\}_{\alpha \geq \beta}$ is a transitive system, which determines a multiplication $\circ$ on $G = \cup_{\alpha \in Y} G_\alpha$, defined by:

$$g_\alpha \circ g_\beta = (g_\alpha \psi_{\alpha,\alpha\beta})(g_\beta \psi_{\beta,\alpha\beta}),$$

and $(G; \circ)$ is a (strong) semilattice Y of the commutative groups, G_α, $\alpha \in Y$. Thus $(G; \circ)$ is a commutative Clifford semigroup. It is clear that Y is the common greatest semilattice homomorphic image of $(G, \circ)$ and $(R, *)$. Moreover, $S = \{(g,r) \in G \times R : \phi_1(g) = \phi_2(r)\}$, where ϕ_1 and ϕ_2 denote the canonical homomorphisms of G and R onto Y, respectively. The proof will be complete if it is shown that, for arbitrary $\alpha, \beta \in Y$ and $(g_\alpha, r_\alpha), (g_\beta, r_\beta) \in S$, the product, $(g_\alpha, r_\alpha)(g_\beta, r_\beta)$ in S equals $(g_\alpha \circ g_\beta, r_\alpha * r_\beta)$.

If $A = (g_\alpha, r_\alpha) \in S_\alpha$ and $B = (g_\beta, r_\beta) \in S_\beta$ are arbitrary elements and $\alpha \geq \beta$, then:

$$AB = ((g_\alpha \psi_{\alpha,\beta}) g_\beta, r_\beta),$$

and:

$$BA = (g_\beta (g_\alpha \psi_{\alpha,\beta}), r_\beta (A \tau_{\alpha,\beta})).$$

Since:

$$\begin{aligned} A^n B A^{n+1} &= (g_\alpha, r_\alpha)^n (g_\beta, r_\beta)(g_\alpha, r_\alpha)^{n+1} \\ &= (g_\alpha^n, r_\alpha)(g_\beta, r_\beta)(g_\alpha^{n+1}, r_\alpha) \\ &= ((g_\alpha^n \psi_{\alpha,\beta}) g_\beta, r_\beta)(g_\alpha^{n+1}, r_\alpha) \\ &= ((g_\alpha^n \psi_{\alpha,\beta}) g_\beta (g_\alpha^{n+1} \psi_{\alpha,\beta}), r_\beta (A^{n+1} \tau_{\alpha,\beta})), \end{aligned}$$

and:

$$\begin{aligned} A^{n+1} B A^n &= (g_\alpha, r_\alpha)^{n+1} (g_\beta, r_\beta)(g_\alpha, r_\alpha)^n \\ &= (g_\alpha^{n+1}, r_\alpha)(g_\beta, r_\beta)(g_\alpha^n, r_\alpha) \\ &= ((g_\alpha^{n+1} \psi_{\alpha,\beta}) g_\beta, r_\beta)(g_\alpha^n, r_\alpha) \\ &= ((g_\alpha^{n+1} \psi_{\alpha,\beta}) g_\beta (g_\alpha^n \psi_{\alpha,\beta}), r_\beta (A^n \tau_{\alpha,\beta})), \end{aligned}$$

one has:

$$A^{n+1} \tau_{\alpha,\beta} = A^n \tau_{\alpha,\beta}.$$

As $\tau_{\alpha,\beta}$ is a homomorphism of S_α into $\mathcal{T}_{R_\beta}$,

$$\begin{aligned} (e_\alpha, r_\alpha)\tau_{\alpha,\beta} &= (g_\alpha^{-n} g_\alpha^n, r_\alpha)\tau_{\alpha,\beta} \\ &= ((g_\alpha^{-n}, r_\alpha)(g_\alpha^n, r_\alpha))\tau_{\alpha,\beta} \\ &= (g_\alpha^{-n}, r_\alpha)\tau_{\alpha,\beta}(g_\alpha^n, r_\alpha)\tau_{\alpha,\beta} \\ &= (g_\alpha^{-n}, r_\alpha)\tau_{\alpha,\beta}(g_\alpha, r_\alpha)^n \tau_{\alpha,\beta} \\ &= (g_\alpha^{-n}, r_\alpha)\tau_{\alpha,\beta} A^n \tau_{\alpha,\beta} \\ &= (g_\alpha^{-n}, r_\alpha)\tau_{\alpha,\beta} A^{n+1} \tau_{\alpha,\beta} \\ &= (g_\alpha^{-n}, r_\alpha)\tau_{\alpha,\beta}(g_\alpha, r_\alpha)^{n+1} \tau_{\alpha,\beta} \\ &= (g_\alpha^{-n}, r_\alpha)\tau_{\alpha,\beta}(g_\alpha^{n+1}, r_\alpha)\tau_{\alpha,\beta} \\ &= ((g_\alpha^{-n}, r_\alpha)(g_\alpha^{n+1}, r_\alpha))\tau_{\alpha,\beta} \\ &= (g_\alpha, r_\alpha)\tau_{\alpha,\beta}. \end{aligned}$$

Thus $(g_\alpha, r_\alpha)\tau_{\alpha,\beta}$ does not depend on g_α and so $\tau_{\alpha,\beta}$ induces a homomorphism $\tau'_{\alpha,\beta}$ of R_α into $\mathcal{T}_{R_\beta}$ defined by:

$$\tau'_{\alpha,\beta} : r_\alpha \mapsto (e_\alpha, r_\alpha)\tau_{\alpha,\beta}.$$

It is noted that if $r_\alpha \in R_\alpha$ and $r_\beta \in R_\beta$ (α and β are arbitrary in Y) and $r_\gamma = r_\alpha * r_\beta$ in R, then;

$$\begin{aligned}(e_\gamma, r_\gamma) &= (e_\alpha, r_\alpha)(e_\beta, r_\beta)\\ &= (e_\alpha, r_\alpha)(e_\beta, r_\beta)(e_\beta, r_\beta)\\ &= (e_\gamma, r_\gamma)(e_\beta, r_\beta)\\ &= (e_\gamma(e_\beta\psi_{\beta,\gamma}), r_\gamma((e_\beta, r_\beta)\tau_{\beta,\gamma}))\\ &= (e_\gamma, r_\gamma(r_\beta\tau'_{\beta,\gamma})),\end{aligned}$$

and so:

$$r_\gamma = r_\gamma(r_\beta\tau'_{\beta,\gamma}),$$

because $\beta \geq \gamma$ and $\psi_{\beta,\gamma}$ maps e_β to e_γ.

Thus, for arbitrary $\alpha, \beta \in Y, (g_\alpha, r_\alpha) \in S_\alpha, (g_\beta, r_\beta) \in S_\beta (\gamma = \alpha\beta$ and $r_\gamma = r_\alpha * r_\beta)$, one has:

$$\begin{aligned}(g_\alpha, r_\alpha)(g_\beta, r_\beta) &= ((g_\alpha\psi_{\alpha,\gamma})(g_\beta\psi_{\beta,\gamma}), r_\gamma((g_\beta, r_\beta)\tau_{\beta,\gamma}))\\ &= ((g_\alpha\psi_{\alpha,\gamma})(g_\beta\psi_{\beta,\gamma}), r_\gamma((e_\beta, r_\beta)\tau_{\beta,\gamma}))\\ &= ((g_\alpha\psi_{\alpha,\gamma})(g_\beta\psi_{\beta,\gamma}), r_\gamma(r_\beta\tau'_{\beta,\gamma}))\\ &= (g_\alpha \circ g_\beta, r_\gamma)\\ &= (g_\alpha \circ g_\beta, r_\alpha * r_\beta).\end{aligned}$$

Thus S is the spined product of the commutative Clifford semigroup $(G; \circ)$ and the right regular band $(R, *)$. Consequently, the first part of the theorem is proved.

If (g_α, r_α) and (g_β, r_β) are arbitrary elements of the spined product S of a commutative Clifford semigroup, $(G, \circ)$ and a right regular band $(R, *)$ $(\alpha, \beta \in Y$, the common greatest semilattice homomorphic image of G and $R)$ then, denoting the identity of G_β by e_β, one has:

$$\begin{aligned}(g_\alpha, r_\alpha)(g_\beta, r_\beta) &= (g_\alpha \circ g_\beta, r_\alpha * r_\beta)\\ &= (g_\alpha \circ g_\beta \circ e_\beta, r_\beta * r_\alpha * r_\beta)\\ &= (g_\beta \circ g_\alpha \circ e_\beta, r_\beta * r_\alpha * r_\beta)\\ &= (g_\beta, r_\beta)(g_\alpha, r_\alpha)(e_\beta, r_\beta),\end{aligned}$$

therefore, S is $\mathcal{R}$-commutative. It is a matter of checking to see that S is also $\mathcal{GC}_n$-commutative (for every n) and regular. Thus the theorem is proved.■

REMARKS

Remark 1

Through this investigation, only the fact that every $\mathcal{GC}_n$-commutative semigroup satisfies the identity $a^n ba^{n+1} = a^{n+1}ba^n$ was used. Thus a regular semigroup is $\mathcal{RGC}_n$-commutative if and only if it is $\mathcal{R}$-commutative and satisfies the identity $a^n ba^{n+1} = a^{n+1}ba^n$.

Remark 2

Theorem 2 shows that a regular semigroup is $\mathcal{RGC}_n$-commutative for some n if and only if it is $\mathcal{RGC}_n$-commutative for every n.

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